

EXTENSIONS OF THE LAURENT DECOMPOSITION AND THE SPACES  $A^p(\Omega)$ 

NIKOLAOS GEORGAKOPOULOS

ABSTRACT. We generalize the classical Laurent decomposition in the setting of domains  $\Omega \subseteq \mathbb{C}$  bounded by Jordan curves. This leads us to study the Fréchet spaces  $A^p(\Omega)$ , and their relation to the spaces  $C^p(\partial\Omega)$ . In the final section, we examine the case of a non Jordan domain  $\Omega$ .

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## 1. INTRODUCTION

It is well known that if  $\Omega \subseteq \mathbb{C}$  is a domain of finite connectivity, then every  $f \in H(\Omega)$  has a decomposition on  $\Omega$  as  $f = f_0 + f_1 + \cdots + f_n$  for  $f_k \in H(V_k)$ , where  $V_1, \dots, V_n$  denote the complements of the connected components of  $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ . This is the Laurent decomposition of  $f$ ; see for instance [5]. If  $f$  extends continuously over  $\overline{\Omega}$ , then it is easily seen that each  $f_k$  extends continuously over  $\overline{V_k}$ . More generally, if  $f \in A^p(\Omega)$  then  $f_k \in A^p(V_k)$ , where for  $p \in \{0, 1, 2, \dots\} \cup \{\infty\}$ ,  $A^p(\Omega)$  denotes the set of functions  $f \in H(\Omega)$  whose derivatives  $f^{(l)}$ ,  $0 \leq l \leq p$ ,  $l \in \mathbb{N}$ , have continuous extensions over  $\overline{\Omega}$ .

As a special case, we can take  $\Omega$  to be a domain bounded by a finite number of pairwise disjoint Jordan curves, such as the annulus  $\Omega = \{z \in \mathbb{C} : r < |z| < 1\}$ ,  $0 < r < 1$ . The limit case as  $r \rightarrow 1^-$  is the case of the unit circle  $\mathbb{T}$ . The Laurent decomposition can be generalized in this setting, as every function  $f \in C^\infty(\mathbb{T})$  can be written as  $f = g + h$  with  $g \in A^\infty(D)$  and  $h \in A^\infty(\mathbb{C} \setminus \overline{D})$ ,  $\lim_{z \rightarrow \infty} h(z) = 0$ ,  $D$  being the open unit disc. The analogous statement for the spaces  $C^p(\mathbb{T})$ ,  $A^p(D)$  and  $A^p(\mathbb{C} \setminus \overline{D})$  does not hold for  $p < +\infty$ ; this is related to the fact that the disc algebra  $A(D) = A^0(D)$  is not complemented in  $C(\mathbb{T}) = C^0(\mathbb{T})$  ([9]). We prove this result in section 3, along with the fact that the spaces  $A^p(D)$  are isomorphic Banach spaces for  $p < +\infty$  (with their natural norms) and the fact that the space  $A^\infty(D)$  is a non normable Fréchet space.

In section 4, we examine potential generalizations of the previous results, when the disc  $D$  is replaced by a Jordan domain  $\Omega$ . The spaces  $C^p(\partial\Omega)$ ,  $0 \leq p \leq +\infty$ , are defined via the parametrization of  $\partial\Omega$  induced by any Riemann mapping of  $D$  onto  $\Omega$ . In order to extend the results of the preceding section, we place additional hypotheses on the geometry of

$\Omega$ , such as the interior chord arc condition ([1]) and the boundedness of the geodesics ([11]), or on the Riemann mapping  $\phi : D \rightarrow \Omega$  itself, such as  $\phi \in A^p(D)$  and  $\phi'(z) \neq 0$  for all  $z \in \overline{D}$ . Under some of those conditions, we also prove that the spaces  $A^p(\Omega)$  and  $A^p(D)$  are isomorphic as Banach spaces, for  $p < +\infty$ , and that  $A^\infty(\Omega)$  has no norm inducing its natural topology.

In section 5, we consider the case of a particular domain  $\Omega$  bounded by two Jordan curves meeting at a point. For  $p = +\infty$  the "Laurent decomposition" is true, but the case  $p < +\infty$  remains open. We suspect the answer is negative, and we have reduced this to the existence of a function  $f \in A(\Omega)$  so that the Cauchy transform of  $f|_{\mathbb{T}}$  ( $\mathbb{T} \subseteq \partial\Omega$ ) diverges as  $z \rightarrow 1$ ,  $|z| < 1$  (see Question 35).

Section 2 contains some preliminary results needed in the sequel.

Finally, we remark that if  $\Omega$  is a domain of finite connectivity, then  $A^p(\Omega)$  is a finite direct sum of spaces  $A^p(V_k)$ , so we can extend our results in this setting, under some additional assumptions on  $\Omega$ . We do not discuss such extensions in the present article, nor do we discuss the several variables case or the density of polynomials or rational functions in  $A^p(\Omega)$ . These facts will be treated elsewhere.

## 2. PRELIMINARIES

We first collect two elementary facts of functional analysis, that we will use throughout this article. The following is well known

**Proposition 1.** *Let  $B$  be a topological vector space and  $A$  be a subspace of  $B$ . The following are equivalent*

1. *There is a continuous linear map  $B \rightarrow A$  that fixes  $A$  (a projection  $B \rightarrow A$ ).*
2. *There is another subspace  $C$  of  $B$  so that  $B = A + C$ ,  $A \cap C = 0$  and the projections  $B \rightarrow A$  and  $B \rightarrow C$  are continuous.*

If the second item holds,  $B$  is the direct sum of  $A$  and  $C$ ,  $B = A \oplus C$ , and  $A$  is complemented in  $B$ . Note that this forces  $A, C$  to be closed subspaces of  $B$ .

**Proposition 2.** *Let  $B$  be a Fréchet space and  $A, C$  be subspaces of  $B$  with  $A + C = B$ ,  $A \cap C = 0$ . The following are equivalent*

1.  $B = A \oplus C$ .
2.  $A$  and  $C$  are closed in  $B$ .
3.  $A$  and  $C$  are Fréchet spaces in the induced topology.

*Proof.* We will prove that condition 3 implies condition 1. If  $A, C$  are Fréchet spaces,  $A \times C$  is a Fréchet space (with the sum of the semi-norms) and the canonical map  $A \times C \rightarrow B$  is a continuous linear bijection. By the Open Mapping Theorem it is an isomorphism, so the projection  $B \rightarrow A \times C \rightarrow A$  is continuous.  $\square$

For  $0 \leq p \leq +\infty$ , we denote by  $A^p(D)$  the space of holomorphic functions on  $D$  whose derivatives of order  $l \in \mathbb{N}$ ,  $0 \leq l \leq p$ , extend continuously over  $\overline{D}$ . It is topologized via the semi-norms

$$|f|_l = \sup_{z \in \overline{D}} |f^{(l)}(z)| = \sup_{z \in \mathbb{T}} |f^{(l)}(z)|, \quad 0 \leq l \leq p, \quad l \in \mathbb{N}$$

$A^p(D)$  is a Banach space for  $p < +\infty$  and a Fréchet space for  $p = +\infty$ . The disc algebra is  $A(D) = A^0(D)$ .

$A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$  is the space of functions  $f$  that are holomorphic on  $D^c$ , whose derivatives of order  $l \in \mathbb{N}$ ,  $0 \leq l \leq p$ , extend continuously over  $\mathbb{C} \setminus D$ , and  $\lim_{z \rightarrow \infty} f(z) = 0$ . Its topology is given by the semi-norms

$$|f|_l = \sup_{z \in D^c} |f^{(l)}(z)| = \sup_{z \in \mathbb{T}} |f^{(l)}(z)|, \quad 0 \leq l \leq p, \quad l \in \mathbb{N}$$

$A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$  is a Banach space for  $p < +\infty$  and a Fréchet space for  $p = +\infty$ . Here,  $\hat{\mathbb{C}}$  denotes the extended plane  $\mathbb{C} \cup \{\infty\}$ .

We will need the following well known fact

**Proposition 3.** *If  $f \in A(D)$  then  $\hat{f}(n) = 0$  for  $n < 0$  and if  $f \in A_0(\hat{\mathbb{C}} \setminus \overline{D})$  then  $\hat{f}(n) = 0$  for  $n \geq 0$ . In both cases, the Laurent coefficients are the Fourier coefficients of the  $2\pi$ -periodic function  $f(e^{i\theta})$ ,  $\theta \in \mathbb{R}$ .*

**Proposition 4.** *Let  $f \in A(D)$  and  $p \geq 1$ . Then  $f \in A^p(D) \iff f|_{\mathbb{T}} \in C^p(\mathbb{T})$ . In that case,*

$$\frac{d^l f}{(de^{i\theta})^l} = f^{(l)}, \quad 1 \leq l \leq p, \quad l \in \mathbb{N}$$

where  $f^{(l)}$  is the continuous extension over  $\mathbb{T}$  of the complex derivative of order  $l$  of  $f$  on  $D$ . The analogous statement is true for  $f \in A(\hat{\mathbb{C}} \setminus \overline{D})$ .

*Proof.* See [7]. □

We note that

$$\frac{df}{d\theta} = \frac{df}{de^{i\theta}} ie^{i\theta}$$

For  $p = 0, 1, \dots, +\infty$ , the topology of  $C^p(\mathbb{T})$  is given by the semi-norms

$$\left\| \frac{d^l f}{d\theta^l} \right\|_{\infty}, \quad 0 \leq l \leq p, \quad l \in \mathbb{N}$$

or equivalently by the semi-norms

$$\left\| \frac{d^l f}{(de^{i\theta})^l} \right\|_{\infty}, \quad 0 \leq l \leq p, \quad l \in \mathbb{N}$$

The topology of  $A^p(D)$  is also induced by the semi-norms

$$\sup_{z \in \mathbb{T}} |f^{(l)}(z)| = \left\| \frac{d^l f}{(de^{i\theta})^l} \right\|_{\infty}$$

It follows that the restriction map  $A^p(D) \rightarrow C^p(\mathbb{T})$  is an embedding, and since the spaces  $A^p(D)$ ,  $C^p(\mathbb{T})$  are complete, we have:

**Proposition 5.** *For all  $p = 0, 1, \dots, +\infty$ ,  $A^p(D)$  and  $A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$  are closed subspaces of  $C^p(\mathbb{T})$ , with trivial intersection.*

### 3. THE CASE OF THE CIRCLE

We begin by showing the analogue of the Laurent decomposition for functions in  $C^\infty(\mathbb{T})$ . To do that, we first collect some well known results regarding the asymptotic behavior of the Fourier coefficients of functions in  $C^\infty(\mathbb{T})$ ,  $A^\infty(D)$  and  $A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$ :

- A function  $f \in C(\mathbb{T})$  is in  $C^\infty(\mathbb{T})$  if and only if  $n^l \hat{f}(n) \rightarrow 0$  as  $|n| \rightarrow +\infty$  for all  $l \geq 0$ .
- A function  $f \in A(D)$  is in  $A^\infty(D)$  if and only if  $n^l \hat{f}(n) \rightarrow 0$  as  $n \rightarrow +\infty$ , for all  $l \geq 0$

- A function  $f \in A_0(\hat{\mathbb{C}} \setminus \overline{D})$  is in  $A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$  if and only if  $n^l \hat{f}(n) \rightarrow 0$  as  $n \rightarrow -\infty$ , for all  $l \geq 0$

It is also easy to see that, if  $f \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$  then  $g(z) = f(1/z)$ ,  $g(0) = 0$ , is in  $A^\infty(D)$ .

The usual topologies of  $C^\infty(\mathbb{T})$ ,  $A^\infty(D)$ ,  $A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$  are also given by these two (equivalent) families of semi-norms:

$$\sup_{n \in \mathbb{Z}} |\hat{f}(n)|, \quad \sup_{n \in \mathbb{Z}} |n^l \hat{f}(n)| \quad \text{for } l > 0, l \in \mathbb{N} \quad (1)$$

$$|\hat{f}(n)| + \sum_{n \in \mathbb{Z}, n \neq 0} |n^l \hat{f}(n)|, \quad l \in \mathbb{N} \quad (2)$$

If  $f \in A^\infty(D)$ , then in (1) and (2) we can have  $n$  range over  $\mathbb{N} = \{0, 1, \dots\}$ , and if  $f \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$  we can have  $n$  range over the negative integers.

**Theorem 6.** Every  $f \in C^\infty(\mathbb{T})$  can be uniquely decomposed as  $f = g + h$  for  $g \in A^\infty(D)$  and  $h \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$ . Moreover,

$$C^\infty(\mathbb{T}) = A^\infty(D) \oplus A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$$

*Proof.* If  $f \in C^\infty(\mathbb{T})$  consider  $g(z) = \sum_{n \geq 0} \hat{f}(n) z^n$ ,  $|z| < 1$ , and  $h(z) = \sum_{n < 0} \hat{f}(n) z^n$ ,  $|z| > 1$ ,  $h(\infty) = 0$ . By the discussion in the beginning of this section,  $g \in A^\infty(D)$ ,  $h \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$ , and of course  $f = g|_{\mathbb{T}} + h|_{\mathbb{T}}$ . The decomposition is unique since  $f = 0 \implies g = -h$  which implies that all Fourier coefficients of  $g, h$  are 0. Finally, we can use Proposition 2 to derive the result regarding the direct sum.  $\square$

There is an equivalent way to state this decomposition:

**Theorem 7.** Every  $f \in C^\infty(\mathbb{T})$  can be uniquely decomposed as  $f = g + \bar{h}$ ,  $g, h \in A^\infty(D)$  and  $h(0) = 0$ .

*Proof.* Let  $f \in C^\infty(\mathbb{T})$ . We have

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n = \sum_{n=0}^{+\infty} a_n z^n + \sum_{n=-\infty}^{-1} a_n z^n, \quad |z| = 1$$

The second summand converges for  $|z| > 1$ , and when restricted to  $|z| = 1$  we have that

$$\overline{\sum_{n=-\infty}^{-1} a_n z^n} = \sum_{n=-\infty}^{-1} \bar{a}_n (\bar{z})^n = \sum_{n=-\infty}^{-1} \bar{a}_n z^{-n} = \sum_{n=1}^{+\infty} \bar{a}_{-n} z^n = h(z)$$

where  $h \in A^\infty(D)$  with  $h(0) = 0$ . So  $f = g + \bar{h}$ , with  $g(z) = \sum_{n=0}^{+\infty} a_n z^n$ . The decomposition is unique, for if  $f = 0$  then  $g = -\bar{h}$  forcing  $g$  and  $h$  to be constant on  $D$ .  $\square$

Consider  $\overline{A_0^\infty(D)} = \{\bar{f} : f \in A^\infty(D), f(0) = 0\}$ , topologized via the semi-norms

$$\|(\bar{f})^{(l)}\|_{\infty, \mathbb{T}}, \quad l \in \mathbb{N}$$

rendering the conjugation map  $\overline{A_0^\infty(D)} \rightarrow A_0^\infty(D)$  an isometric isomorphism. We embed  $\overline{A_0^\infty(D)}$  into  $C^\infty(\mathbb{T})$  via the map  $f \mapsto f|_{\mathbb{T}}$ , and under this embedding, we have

$$\overline{A_0^\infty(D)} = A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$$

This was shown in the proof of Theorem 7. By Theorem 6 we obtain,

**Corollary 8.**  $C^\infty(\mathbb{T}) = A^\infty(D) \oplus \overline{A_0^\infty(D)}$ .

In the rest of this section, we shall prove that similar decompositions are impossible for  $p < +\infty$ . We begin with some useful isomorphisms, the first of which was communicated to us by C. Panagiotis.

**Theorem 9.** *For all  $p < +\infty$ , there is an isomorphism  $\Phi : C^{p+1}(\mathbb{T}) \rightarrow C^p(\mathbb{T})$  that restricts to isomorphisms  $A^{p+1}(D) \rightarrow A^p(D)$  and  $A_0^{p+1}(\hat{\mathbb{C}} \setminus \overline{D}) \rightarrow A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$ .*

*It follows that the spaces  $C^0(\mathbb{T}), C^1(\mathbb{T}), C^2(\mathbb{T}), \dots$  are isomorphic Banach spaces. The same is true for the spaces  $A^0(D), A^1(D), A^2(D), \dots$ .*

*Proof.* Let  $\Phi : C^{p+1}(\mathbb{T}) \rightarrow C^p(\mathbb{T})$  be given by

$$\Phi(f) = \frac{df}{d\theta} + \hat{f}(0) \quad (3)$$

This map is clearly continuous and linear, and it is easy to see that it is injective (a linear function  $a\theta + b$ ,  $a, b \in \mathbb{C}$ , must be  $2\pi$ -periodic hence constant). We shall now show surjectivity. Let  $g \in C^p(\mathbb{T})$  and consider the function  $S_g \in C^{p+1}(\mathbb{T})$  given by,

$$S_g(e^{i\theta}) = \int_0^\theta g(e^{i\omega}) d\omega - \hat{g}(0)\theta$$

If  $f \in C^{p+1}(\mathbb{T})$  is given by

$$f = S_g - \widehat{S_g}(0) + \hat{g}(0) \quad (4)$$

then we can easily check that  $g = \Phi(f)$ , so  $\Phi$  is surjective. By the Open Mapping Theorem,  $\Phi : C^{p+1}(\mathbb{T}) \rightarrow C^p(\mathbb{T})$  is an isomorphism, and its inverse is given by the assignment  $g \mapsto f$ ,  $f$  being as in (4).

We shall now prove that  $\Phi$  restricts to an isomorphism  $A^{p+1}(D) \rightarrow A^p(D)$ . If  $f \in A^{p+1}(D)$ , we have

$$\Phi(f)(z) = \frac{df}{d\theta}(z) + \hat{f}(0) = \frac{df}{de^{i\theta}}(z)ie^{i\theta} + f(0) = f'(z)zi + f(0)$$

for  $z = e^{i\theta}$ . This allows us to extend  $\Phi(f)$  over the closed disc. Therefore, we have the map  $\Phi : A^{p+1}(D) \rightarrow A^p(D)$ ,

$$\Phi(f)(z) = f'(z)zi + f(0) \quad (5)$$

$\Phi$  is clearly a continuous linear injection  $A^{p+1}(D) \rightarrow A^p(D)$ . Before we show surjectivity, let us note that every function in  $A^{p+1}(D)$  has an antiderivative in  $A^p(D)$ . Indeed,

$$F(z) = \int_{[0,z]} f(\zeta) d\zeta$$

is an antiderivative of  $f$  on  $D$ , and it is easy to check the uniform continuity of  $F$  over  $D$ , hence  $F$  extends continuously over  $\overline{D}$ . The first  $p$  derivatives of  $F$  also extend continuously over  $\overline{D}$ , so  $F \in A^{p+1}(D)$ . To show that  $\Phi : A^{p+1}(D) \rightarrow A^p(D)$  is surjective, let  $g \in A^p(D)$ . The function

$$r(z) = \begin{cases} \frac{g(z) - g(0)}{z} & \text{if, } z \neq 0 \\ g'(0) & \text{if, } z = 0 \end{cases}$$

has an antiderivative  $G$  with  $G(0) = 0$ . Consider  $f : \overline{D} \rightarrow \mathbb{C}$ ,

$$f(z) = -iG(z) + g(0) \quad (6)$$

One can easily check that  $f \in A^{p+1}(D)$  and  $\Phi(f) = g$ , so  $\Phi : A^{p+1}(D) \rightarrow A^p(D)$  is also surjective. By the Open Mapping Theorem,  $\Phi$  is an isomorphism, with inverse given by  $g \mapsto f$ ,  $f$  as in (6).

Finally, let us show that the map  $\Phi$  given in (3) restricts to an isomorphism  $A_0^{p+1}(\hat{\mathbb{C}} \setminus \overline{D}) \rightarrow A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$ . If  $f \in A_0^{p+1}(\hat{\mathbb{C}} \setminus \overline{D})$  then  $g(z) = f(1/z)$  is in  $A^{p+1}(D)$  and  $g(0) = 0$ . Then,  $\Phi(g)$  is in  $A^p(D)$  and is given by

$$\Phi(g)(z) = (f(z^{-1}))'zi + f(\infty) = f'(z^{-1})(-iz^{-1})$$

The function  $h : D^c \rightarrow \mathbb{C}$  given by  $h(z) = -\Phi(g)(1/z)$  is in  $A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$  and

$$h(z) = f'(z)iz$$

The assignment  $f \mapsto h$  defines an isomorphism  $A_0^{p+1}(\hat{\mathbb{C}} \setminus \overline{D}) \rightarrow A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$  as it is the composition of isomorphisms  $f \mapsto g \mapsto \Phi(g) \mapsto h$ . Note that if we restrict  $h(z) = f'(z)iz$  on  $\mathbb{T}$  we have  $h|_{\mathbb{T}} = \Phi(f)$ . Therefore,  $\Phi : C^{p+1}(\mathbb{T}) \rightarrow C^p(\mathbb{T})$  restricts to an isomorphism  $A_0^{p+1}(\hat{\mathbb{C}} \setminus \overline{D}) \rightarrow A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$ . This completes the proof.  $\square$

**Theorem 10.** *If  $p < +\infty$ , there is an  $f \in C^p(\mathbb{T})$  that can't be written as  $f = g + h$  for any  $g \in A^p(D)$  and any  $h \in A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$ .*

*Proof.* We induct on  $p$ . The base case  $p = 0$  can be found in [9] (compare with Proposition 2). So assume that  $C^p(\mathbb{T}) \neq A^p(D) + A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$  while  $C^{p+1}(\mathbb{T}) = A^{p+1}(D) + A_0^{p+1}(\hat{\mathbb{C}} \setminus \overline{D})$  and take an  $f \in C^p(\mathbb{T})$  not in  $A^p(D) + A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$ . If  $\Phi$  is as in Theorem 9,  $\Phi^{-1}(f) \in C^{p+1}(\mathbb{T})$  hence  $\Phi^{-1}(f) = g + h$  for some  $g \in A^{p+1}(D)$  and  $h \in A_0^{p+1}(\hat{\mathbb{C}} \setminus \overline{D})$ . But then  $f = \Phi(g) + \Phi(h)$  with  $\Phi(g) \in A^p(D)$  and  $\Phi(h) \in A_0^p(\hat{\mathbb{C}} \setminus \overline{D})$ , contradicting our assumption on  $f$ .  $\square$

We can actually prove something quite stronger (compare with Proposition 2):

**Theorem 11.** *If  $p < +\infty$ ,  $A^p(D)$  is not isomorphic to any complemented subspace of  $C^p(\mathbb{T})$ .*

*Proof.* The case  $p = 0$  can be found in [12]. We induct on  $p$ , and assume that  $A^p(D)$  is not isomorphic to any complemented subspace of  $C^p(\mathbb{T})$ . If there are closed subspaces  $K, L$  of  $C^{p+1}(\mathbb{T})$  such that  $C^{p+1}(\mathbb{T}) = K \oplus L$  and  $K \approx A^{p+1}(D)$  (we use the symbol  $\approx$  for isomorphisms), then applying the isomorphism  $\Phi$  of Theorem 9 on  $C^{p+1}(\mathbb{T}) = K \oplus L$ , we obtain that  $C^p(\mathbb{T}) = K' \oplus L'$  for  $K' \approx K$  and  $L' \approx L$ . Therefore,  $A^p(D) \approx A^{p+1}(D) \approx K \approx K'$  (the first isomorphism is by Theorem 9) hence  $A^p(D)$  is isomorphic to a complemented subspace of  $C^p(\mathbb{T})$ , contradicting the induction hypothesis.  $\square$

As we showed in Theorem 9, all spaces  $C^p(\mathbb{T})$  are isomorphic for  $p < +\infty$ , and all spaces  $A^p(D)$  are isomorphic for  $p < +\infty$ . This fails if we allow  $p = +\infty$ :

**Theorem 12.** *There is no norm inducing the usual topologies on  $C^\infty(\mathbb{T})$  and  $A^\infty(D)$ . Therefore,  $C^\infty(\mathbb{T})$  is not isomorphic to  $C(\mathbb{T}) = C^0(\mathbb{T})$  and  $A^\infty(D)$  is not isomorphic to  $A(D) = A^0(D)$ .*

*Proof.* We will prove that if  $C^\infty(\mathbb{T})$  had a norm  $\|\cdot\|$  inducing the same topology as the semi-norms  $|\cdot|_l$ , then its closed unit ball would be compact, contradicting that it is an infinite dimensional Banach space. Let  $\|f_n\| \leq 1$  and fix  $l$  momentarily. The set

$$\{f \in C^\infty(\mathbb{T}) : |f|_l < 1\}$$

is an open neighborhood of 0, so it contains some

$$\{f \in C^\infty(\mathbb{T}) : \|f\| < r\}$$

Thus,  $|rf_n/2|_l < 1 \implies |f_n|_l < M_l$  uniformly for all  $n$ , for some  $M_l < +\infty$ . So it remains to show that  $f_n$  has a subsequence converging to some  $f \in C^\infty$  in all semi-norms  $|\cdot|_l$ .

The sequence  $f_n$  is a uniformly bounded ( $\|f_n\|_\infty = |f_n|_0 < M_0$ ) and equicontinuous ( $\|f'_n\|_\infty = |f_n|_1 < M_1$ ) family of continuous functions on the compact set  $\mathbb{T}$ , so by the Arzelà-Ascoli theorem, it has a subsequence  $f_{k_n,1}$  converging to some  $f \in C(\mathbb{T})$  uniformly. Similarly,  $f_{k_n,1}$  has a subsequence  $f'_{k_n,2}$  converging to some  $g \in C(\mathbb{T})$  uniformly; it follows that  $f' = g$ . Iterating shows that  $f \in C^\infty(\mathbb{T})$  and  $f_{k_n,n} \rightarrow f$  in every semi-norm. The unit ball is thus compact, and we have our contradiction.

This argument can be adapted for  $A^\infty(D)$  as follows: Let  $f_n \in A^\infty(D)$  and  $|f_n|_l < M_l < +\infty$  uniformly on  $n$ , and for all  $l$ . The sequence  $f_n \in C(\bar{D})$  is uniformly bounded and equicontinuous ( $f'_n$  is uniformly bounded and  $D$  is convex) so it has a subsequence  $f_{k_n,1}$  converging uniformly to a continuous function  $f$  on  $\bar{D}$ . The function is then holomorphic on  $D$  hence  $f \in A(D)$ . The same argument gives a subsequence  $f_{k_n,2} \in C(\bar{D})$  with  $f'_{k_n,2} \rightarrow g$  uniformly on  $\bar{D}$ ; obviously  $f' = g$  on  $D$  so  $g$  is the continuous extension of  $f'$  on  $\mathbb{T}$ . Thus,  $f_{k_n,2} \rightarrow f$  in the topology of  $A^1(D)$ . Iterating shows that  $f \in A^\infty(D)$  and  $f_{k_n,n} \rightarrow f$  in every semi-norm defining the topology of  $A^\infty(D)$ . This completes the proof.  $\square$

#### 4. JORDAN DOMAIN

In this section,  $\Omega \subseteq \mathbb{C}$  is a fixed Jordan domain (a simply connected region whose boundary  $\partial\Omega$  is a Jordan curve) and  $\phi : D \rightarrow \Omega$  is a fixed Riemann map.

By the Carathéodory-Osgood Theorem,  $\phi$  extends to a homeomorphism  $\phi : \bar{D} \rightarrow \bar{\Omega}$ . The homeomorphism  $\gamma : \mathbb{T} \rightarrow \partial\Omega$ ,  $\gamma = \phi|_{\mathbb{T}}$ , is the parameterization of  $\partial\Omega$  that we will be using.

**Definition 13.** We define  $C^p(\partial\Omega)$  as the space of functions  $f : \partial\Omega \rightarrow \mathbb{C}$  such that  $f \circ \gamma \in C^p(\mathbb{T})$ , and endow it with semi-norms

$$\left\| \frac{d^l f}{d\theta^l} \right\|_\infty = \left\| \frac{d^l (f \circ \gamma)}{d\theta^l} \right\|_{\infty, \mathbb{T}}, \quad 0 \leq l \leq p, \quad l \in \mathbb{N}$$

It is a Fréchet space for all  $p$  and a Banach space for  $p < +\infty$ . By definition,  $C^p(\partial\Omega)$  is isometrically isomorphic to  $C^p(\mathbb{T})$ , via the map  $f \mapsto f \circ \gamma$ .

$A^p(\Omega)$  is defined as the space of holomorphic functions  $f$  on  $\Omega$ , whose derivatives  $f^{(l)}$ ,  $0 \leq l \leq p$ ,  $l \in \mathbb{N}$ , extend continuously over  $\bar{\Omega}$ . Its topology is defined by the semi-norms

$$\|f^{(l)}\|_{\infty, \Omega} = \|f^{(l)}\|_{\infty, \partial\Omega}, \quad 0 \leq l \leq p, \quad l \in \mathbb{N}$$

and it is a Fréchet space for all  $p$  and a Banach space for  $p < +\infty$ . By definition,  $A(\Omega) = A^0(\Omega)$  is isometrically isomorphic to  $A(D)$ , via the map  $f \mapsto f \circ \phi$ . In Theorem 21 below, we shall give sufficient conditions under which  $A^p(\Omega) \approx A^p(D)$  for  $0 < p < +\infty$ . The case  $p = +\infty$  is open; see Question 23 below.

We define  $A_0^p(\hat{\mathbb{C}} \setminus \bar{\Omega})$  analogously, with the additional condition that  $\lim_{z \rightarrow \infty} f(z) = 0$ .

Finally, we define

$$\overline{A_0^p(\Omega)} = \{\bar{f} : f \in A^p(\Omega), f(\phi(0)) = 0\}$$

is topologized by semi-norms

$$\|(\bar{f})^{(l)}\|_{\infty, \Omega} = \|(\bar{f})^{(l)}\|_{\infty, \partial\Omega}, \quad 0 \leq l \leq p, \quad l \in \mathbb{N}$$

and the conjugation map  $\overline{A_0^p(\Omega)} \rightarrow A_0^p(\Omega)$  becomes an isometric isomorphism.

The decomposition in Theorem 6 is the trickiest to generalize in this setting, so we postpone this until the end of this section, and instead proceed with extending the decomposition in Theorem 7:

**Theorem 14.** *If the Riemann map  $\phi : D \rightarrow \Omega$  is in  $A^\infty(D)$  and  $\phi'(z) \neq 0$  for all  $z \in \mathbb{T}$ , then every  $f \in C^\infty(\partial\Omega)$  has a unique decomposition as  $f = g + \bar{h}$  with  $g, h \in A^\infty(\Omega)$  and  $h(\phi(0)) = 0$ .*

*Proof.* If  $f \in C^\infty(\partial\Omega)$  then  $f \circ \gamma \in C^\infty(\mathbb{T})$  can be decomposed as  $f \circ \gamma = g + \bar{h}$ , for  $g, h \in A^\infty(D)$  and  $h(0) = 0$ , by Theorem 7. So

$$f = g \circ \gamma^{-1} + \bar{h} \circ \gamma^{-1} = g \circ \gamma^{-1} + \overline{h \circ \gamma^{-1}}$$

and  $g \circ \gamma^{-1}, h \circ \gamma^{-1} \in A^\infty(\Omega)$  (their analytic extensions over  $\Omega$  are  $g \circ \phi^{-1}$  and  $h \circ \phi^{-1}$ ; we have  $\phi^{-1} \in A^\infty(\Omega)$  because  $\phi \in A^\infty(\Omega)$  and  $\phi' \neq 0$ ) and  $(h \circ \phi^{-1})(\phi(0)) = h(0) = 0$ .

The uniqueness of the decomposition  $f = g + \bar{h}$  follows from the fact if  $f = 0$  then  $g = \bar{h}$  for holomorphic  $g, h$  on a domain, hence both  $g$  and  $h$  are constant.  $\square$

We now examine when  $A^\infty(\Omega)$  embeds in  $C^\infty(\partial\Omega)$  in the canonical way, and more generally when  $A^p(\Omega)$  embeds in  $C^p(\partial\Omega)$ ,  $0 \leq p \leq +\infty$ :

**Theorem 15.** *Let  $0 \leq p \leq +\infty$  be fixed. The following are equivalent*

1. *The map  $A^p(\Omega) \rightarrow C^p(\partial\Omega)$  given by restriction to the boundary is well defined; that is,  $f|_{\partial\Omega} \in C^p(\partial\Omega)$  for every  $f \in A^p(\Omega)$ .*
2.  *$\gamma \in C^p(\mathbb{T})$ .*
3.  *$\phi \in A^p(D)$ .*

*Proof.* This is trivial for  $p = 0$  (by continuity), so assume  $p \geq 1$ . The equivalence of items 2 and 3 is part of Proposition 4 (see [7]).

If the restriction map  $A^p(\Omega) \rightarrow C^p(\partial\Omega)$  is well defined, then  $\text{id}_\Omega \in A^p(\Omega)$  would restrict to  $\text{id}_{\partial\Omega} \in C^p(\partial\Omega)$ , namely  $\text{id}_{\partial\Omega} \circ \gamma \in C^p(\mathbb{T}) \iff \gamma \in C^p(\mathbb{T})$ .

For the converse, first take  $p = 1$  and let  $f \in A^1(\Omega)$  and  $\tilde{f}, \tilde{f}'$  be the extensions of  $f, f'$  over the boundary. We want to show that  $\tilde{f} \circ \gamma$  is  $C^1$  smooth.

For  $0 \leq r < 1$  set  $\gamma_r(e^{i\theta}) = \phi(re^{i\theta})$ . For convenience, we denote  $u' = \frac{du}{d\theta}$  for  $u \in C^1(\mathbb{T})$ . By the uniform continuity of  $\phi$  we have that  $\gamma_r \rightarrow \gamma$  uniformly. In addition,  $\gamma'_r \rightarrow \gamma'$  uniformly, since for  $r < 1$  we have

$$\begin{aligned} |\gamma'_r(e^{i\theta}) - \gamma'(e^{i\theta})| &= |\phi'(re^{i\theta})r - \phi'(e^{i\theta})| \leq \\ &= |\phi'(re^{i\theta})||r - 1| + |\phi'(re^{i\theta}) - \phi'(e^{i\theta})| \leq \\ &= \|\phi'\|_\infty(1 - r) + |\phi'(re^{i\theta}) - \phi'(e^{i\theta})| \end{aligned}$$

which is arbitrarily small for  $r$  sufficiently close to 1, by the uniform continuity of  $\phi'$ . So  $f \circ \gamma_r = \tilde{f} \circ \gamma_r \rightarrow \tilde{f} \circ \gamma$  uniformly, and  $(f \circ \gamma_r)' = (f' \circ \gamma_r)\gamma'_r \rightarrow (\tilde{f}' \circ \gamma)\gamma'$  uniformly. By a theorem in real analysis, we conclude that  $\tilde{f} \circ \gamma$  is differentiable and

$$\frac{d\tilde{f}}{d\theta} = (\tilde{f}' \circ \gamma)\gamma' \tag{7}$$

An induction on  $p$  completes the proof.  $\square$

So under the assumption  $\gamma \in C^p(\mathbb{T})$ , we have  $A^p(\Omega)$  as a subset of  $C^p(\mathbb{T})$ . To have it embedded in  $C^p(\mathbb{T})$  as a closed subspace, we additionally need the usual topology on  $A^p(\Omega)$  given by the semi-norms

$$\|f^{(l)}\|_{\infty, \Omega} = \|f^{(l)}\|_{\infty, \partial\Omega}, \quad 0 \leq l \leq p, \quad l \in \mathbb{N}$$



to agree with the relative topology induced by  $C^p(\mathbb{T})$ , i.e., given by the semi-norms

$$\left\| \frac{d^l(f \circ \gamma)}{d\theta^l} \right\|_{\infty, \mathbb{T}}, 0 \leq l \leq p, l \in \mathbb{N}$$

A sufficient condition is that  $\gamma' \neq 0$  (for  $p \geq 1$  of course). Indeed, by the equations

$$\frac{df}{d\theta} = \frac{df}{dz} \gamma' \iff \frac{df}{dz} = \frac{df}{d\theta} (\gamma')^{-1}$$

we can prove that the two semi-norms for  $l = 1$  are equivalent (they are equal for  $l = 0$ ). This can be done for all  $l \leq p, l \in \mathbb{N}$ , by induction, see for instance [4].

**Theorem 16.** *Let  $p \geq 1$ . If  $\gamma \in C^p(\mathbb{T})$  and  $\gamma' \neq 0$  then  $A^p(\Omega)$  is a closed subspace of  $C^p(\partial\Omega)$ . For  $p = 0$ , we always have  $A(\Omega) = A^0(\Omega)$  as a closed subspace of  $C(\partial\Omega) = C^0(\partial\Omega)$ .*

*Proof.* The image of  $A^p(\Omega)$  in  $C^p(\partial\Omega)$  under the restriction map,  $f \mapsto f|_{\partial\Omega}$ , is isomorphic to the complete space  $A^p(\Omega)$  by the preceding discussion. Therefore,  $A^p(\Omega)$  is a closed subspace of  $C^p(\partial\Omega)$   $\square$

The analogous statement is true for  $\overline{A^p(\Omega)}$ . Therefore, by Theorem 14 and Proposition 2, we have that

**Corollary 17.** *If  $\gamma \in C^\infty(\mathbb{T})$  and  $\gamma' \neq 0$  then  $C^\infty(\partial\Omega) = A^\infty(\Omega) \oplus \overline{A_0^\infty(\Omega)}$ .*

We shall prove that no such decomposition can occur for  $p < +\infty$ , under some mild conditions on  $\Omega$ . Precisely, we define the family of Jordan domains  $\mathcal{F}$  consisting of all  $\Omega$  with one of the following properties:

1.  $\Omega$  has rectifiable boundary.
2.  $\Omega$  is star-like.
3.  $\Omega$  is bounded by the graph of a continuous function  $s : [0, 1] \rightarrow \mathbb{R}$  and the horizontal axis positioned at height  $c < \min s$ :

$$\Omega = \{(x, y) \in \mathbb{C} : 0 < x < 1, c < y < s(x)\}$$

4. For any  $\epsilon > 0$  there is a  $\delta > 0$  so that whenever  $z, w \in \Omega$ ,  $|z - w| < \delta$ , there is a curve in  $\Omega$  connecting them with length less than  $\epsilon$ .
5. There is a  $C > 0$  such that for any two points  $z, w \in \Omega$  there is a curve in  $\Omega$  connecting them with length less than  $C|z - w|$ . This is the so called "interior chord arc condition" ([1]).
6. The Riemann map  $\phi : D \rightarrow \Omega$  is Lipschitz continuous. This is equivalent to Lipschitz continuity on  $\overline{D}$ , and also equivalent to  $\phi'$  being bounded on  $D$  (because  $D$  is convex).

We remark that each one of conditions 5 and 6 imply condition 4 (if  $\phi$  is Lipschitz then we can use curves  $\phi([\phi^{-1}(z), \phi^{-1}(w)])$  to connect  $z, w$ ), while condition 6 clearly implies condition 1.

**Question 18.** Do conditions 4 or 5 imply condition 1?

**Theorem 19.** *If  $\Omega \in \mathcal{F}$  then every function in  $A(\Omega)$  has an antiderivative in  $A^1(\Omega)$ ; in other words, the integration operator on  $\Omega$  maps  $A(\Omega)$  to itself.*

*Proof.* We refer the reader to [8] for the cases of conditions 1,2 and 3. We shall only treat condition 4, which is weaker than conditions 5 and 6.

Assume that for any  $\epsilon > 0$  there is a  $\delta > 0$  such that whenever  $z, w \in \Omega$ ,  $|z - w| < \delta$ , then there is a rectifiable curve  $\delta_{z,w}$  in  $\Omega$  connecting  $z, w$  with length  $\ell(\delta_{z,w})$  less than  $\epsilon$ . For arbitrary curves  $\gamma_z$  in  $\Omega$  starting from a fixed  $z_0 \in \Omega$  to  $z$ , the function  $F : D \rightarrow \mathbb{C}$  given by  $F(z) = \int_{\gamma_z} f(\zeta) d\zeta$  is an antiderivative of  $f$  in  $\Omega$ , and is independent of the choice of curves  $\gamma_z$ . For  $z, w \in \Omega$  and  $|z - w| < \delta$  we have

$$|F(z) - F(w)| = \left| \int_{\delta_{z,w}} f(\zeta) d\zeta \right| \leq \|f\|_{\infty} \ell(\delta_{z,w}) < \|f\|_{\infty} \epsilon$$

which can become arbitrarily small as  $\|f\|_{\infty} < +\infty$ . Therefore,  $F$  is uniformly continuous on  $\Omega$ , hence extends continuously over  $\overline{\Omega}$ . Because  $F' = f \in A(\Omega)$ , we conclude that  $F \in A^1(\Omega)$ .  $\square$

**Remark 20.** This proof also yields that if  $\Omega$  has property 4, then every function in  $H^{\infty}(\Omega)$  (a function holomorphic and bounded on  $\Omega$ ) has an antiderivative in  $A(\Omega)$ . The integration operator maps  $H^{\infty}(\Omega)$  to itself if and only if there is an  $M \in (0, +\infty)$  so that any two points in  $\Omega$  can be joined by a curve in  $\Omega$  with length at most  $M$  ([11]).

**Theorem 21.** *If the integration operator on  $\Omega$  maps  $A(\Omega)$  to itself and  $p < +\infty$ , then  $A^p(\Omega) \approx A^{p+1}(\Omega)$  and  $A^p(\Omega) \approx A^p(D)$ . In particular, the conclusion holds for  $\Omega \in \mathcal{F}$ .*

*Proof.* The first statement follows exactly as in the case of the disc: After a translation, we may assume  $0 \in \Omega^{\circ}$ . The map  $\Phi : A^{p+1}(\Omega) \rightarrow A^p(\Omega)$  given by

$$\Phi(f)(z) = f'(z)zi + f(0)$$

is an isomorphism, the proof of which is similar to the second part of the proof of Theorem 9. In addition,  $A^p(\Omega) \approx A(\Omega) \approx A(D) \approx A^p(D)$  where the second isomorphism is given by  $f \mapsto f \circ \phi$ .  $\square$

As  $C^{\infty}(\mathbb{T}) \approx C^{\infty}(\partial\Omega)$ ,  $C^{\infty}(\partial\Omega)$  has no norm inducing its usual topology by Theorem 12.

**Theorem 22.**  *$A^{\infty}(\Omega)$  has no norm inducing its usual topology, if  $\Omega$  has property 4 (or properties 5,6).*

*Proof.* This is a straightforward adaptation of the proof of Theorem 12. The only difference is in showing that uniform boundedness of  $f'_n$  on  $\overline{\Omega}$  implies equicontinuity of  $f_n$  on  $\overline{\Omega}$ . Here is where the condition 4 comes into play: For arbitrary  $\epsilon > 0$  take  $\delta > 0$  so that any two  $z, w \in \Omega$  less than  $\delta$  apart can be connected by a curve  $\delta_{z,w}$  in  $\Omega$  with length less than  $\epsilon/M$ ,  $M$  being the uniform bound on  $f'_n$ . Then,

$$|f_n(z) - f_n(w)| = \left| \int_{\delta_{z,w}} f'_n(\zeta) d\zeta \right| \leq M \ell(\delta_{z,w}) < \epsilon$$

where by  $\ell(\delta_{z,w})$  we denote the length of the curve  $\delta_{z,w}$ . So we have uniform equicontinuity on  $\Omega$ , which in turn implies equicontinuity on  $\overline{\Omega}$ . The rest of the proof is similar to the proof of Theorem 12.  $\square$

**Question 23.** Under what assumptions on  $\Omega$  are the spaces  $A^{\infty}(D)$  and  $A^{\infty}(\Omega)$  isomorphic?

We now generalize Theorem 11

**Theorem 24.** *If either  $p = 0$  or  $1 \leq p < +\infty$  and the integration operator maps  $A(\Omega)$  to itself, then  $A^p(\Omega)$  is not isomorphic to any complemented subspace of  $C^p(\partial\Omega)$ . In particular, this holds for  $\Omega \in \mathcal{F}$  and  $p < +\infty$ .*

*Proof.* First take  $p = 0$  and assume there are  $K, L$  so that  $C(\partial\Omega) = K \oplus L$  and  $A(\Omega) \approx K$ . We apply the isomorphism  $C(\partial\Omega) \rightarrow C(\mathbb{T})$  (given by  $f \mapsto f \circ \gamma$ ) on  $C(\partial\Omega) = K \oplus L$  to obtain that  $C(\mathbb{T}) = K' \oplus L'$  for  $K', L'$  isomorphic to  $K, L$  respectively. But then  $K' \approx K \approx A(\Omega) \approx A(D)$  and  $K'$  is complemented in  $C(\mathbb{T})$ , contradicting Theorem 11 ([12]).

Now take  $\Omega \in \mathcal{F}$ ,  $p < +\infty$ , and assume that  $C^p(\partial\Omega) = K \oplus L$  for some  $K, L$  with  $A^p(\Omega) \approx K$ . As before, we have  $C^p(\mathbb{T}) \approx K' \oplus L'$  for  $K' \approx K$ . The space  $K$  is isomorphic to  $A^p(\Omega)$ , which by Theorem 21 is isomorphic to  $A^p(D)$ ; consequently,  $A^p(D)$  is isomorphic to the complemented subspace  $K'$  of  $C^p(\mathbb{T})$ , contradicting Theorem 11.  $\square$

**Proposition 25.** *If  $\partial\Omega$  has continuous analytic capacity 0 then  $A^p(\Omega) \cap A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega})$  is trivial for all  $p$ .*

*Proof.* A function  $f$  in the intersection of the two spaces would have to be continuous on  $\mathbb{C}$  and holomorphic on  $\mathbb{C} \setminus \partial\Omega$ . The boundary  $\partial\Omega$  having zero continuous analytic capacity translates to the fact that such  $f$  is entire ([4], [6]). Since  $\lim_{z \rightarrow \infty} f(z) = 0$  we have by Liouville's Theorem that  $f$  is identically zero.  $\square$

The complement of  $\overline{\Omega}$  in the Riemann sphere is simply connected, so there is a Riemann map  $\psi : \overline{D}^c \rightarrow \overline{\Omega}^c$ . By the Carathéodory-Osgood Theorem, the map  $\psi$  extends over the boundaries  $\partial D, \partial\Omega$ , as a homeomorphism  $\delta : \mathbb{T} \rightarrow \partial\Omega$ . The composition  $\delta \circ \gamma^{-1}$  is a homeomorphism of  $\partial\Omega$ , and  $\gamma^{-1} \circ \delta$  is a homeomorphism of  $\mathbb{T}$ . As the only injective entire maps  $\mathbb{C} \rightarrow \mathbb{C}$  are linear,  $\gamma \neq \delta$  unless  $\Omega$  is a disc in the plane; the homeomorphism  $\gamma^{-1} \circ \delta : \mathbb{T} \rightarrow \mathbb{T}$  is usually not the identity mapping. Any homeomorphism of  $\mathbb{T}$  obtained this way (for arbitrary Jordan domain  $\Omega$ ) is called a welding ([2], [3]).

**Proposition 26.** *If  $\delta \in C^p(\mathbb{T})$  and  $\delta' \neq 0$  then  $A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega})$  is a closed subspace of  $C^p(\partial\Omega)$ , embedded via the restriction map.*

*Proof.* Analogous to the proof of Theorem 16.  $\square$

**Theorem 27.** *If either one of the following items are true*

- $p = 0$  and  $\partial\Omega$  has continuous analytic capacity 0
- $1 \leq p < +\infty$  and  $\gamma, \delta \in C^p(\mathbb{T})$ ,  $\gamma', \delta' \neq 0$

*then there is a function in  $C^p(\partial\Omega)$  that can't be decomposed as the sum of a function in  $A^p(\Omega)$  and another in  $A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega})$ .*

*Proof.* Under our conditions,  $A^p(\Omega)$  and  $A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega})$  are closed subspaces of  $C^p(\partial\Omega)$  (Propositions 16 and 26) with trivial intersection (Proposition 25). Therefore, by Proposition 2, if we assume that  $C^p(\partial\Omega) = A^p(\Omega) + A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega})$  then  $A^p(\Omega)$  is complemented in  $C^p(\partial\Omega)$ , contradicting Theorem 24. Thus,  $C^p(\partial\Omega) \neq A^p(\Omega) + A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega})$ .  $\square$

We shall now describe some other splittings of  $C^\infty(\mathbb{T})$  in terms of  $A^\infty(\Omega)$ ,  $A_0^\infty(\hat{\mathbb{C}} \setminus \overline{\Omega})$  and the welding  $\delta \circ \gamma^{-1}$  (and its inverse). But before we do that, let us fix some notation:

**Notation 28.** If  $A$  is a set of functions  $X \rightarrow Y$  and  $g : Z \rightarrow X$  then we denote  $A \circ g = \{f \circ g : f \in A\}$ .

**Proposition 29.** Assume  $\gamma, \delta \in C^\infty(\mathbb{T})$  and  $\gamma', \delta' \neq 0$ . Then

- $C^\infty(\partial\Omega) = A^\infty(\Omega) \oplus [A_0^\infty(\hat{\mathbb{C}} \setminus \overline{\Omega}) \circ \delta \circ \gamma^{-1}]$
- $C^\infty(\partial\Omega) = [A^\infty(\Omega) \circ \gamma \circ \delta^{-1}] \oplus A_0^\infty(\hat{\mathbb{C}} \setminus \overline{\Omega})$

*Proof.* Let  $P : C^\infty(\mathbb{T}) \rightarrow A^\infty(D)$  be the canonical projection,  $P(\sum_{n=-\infty}^{+\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n z^n$ . Then  $Q : C^\infty(\partial\Omega) \rightarrow A^\infty(\Omega)$  defined by  $Q(f) = P(f \circ \gamma) \circ \gamma^{-1}$  is a projection (it fixes  $A^\infty(\Omega)$  precisely because  $P$  does). Therefore,  $C^\infty(\partial\Omega) = A^\infty(\Omega) \oplus \text{Ker} Q$ . We determine the kernel:

$$\begin{aligned} Q(f) = 0 &\iff P(f \circ \gamma) = 0 \iff \\ &f \circ \gamma \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D}) \iff f \circ \gamma \circ \delta^{-1} \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{\Omega}) \iff \\ &f \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{\Omega}) \circ \delta \circ \gamma^{-1} \end{aligned}$$

The other item follows similarly.  $\square$

We shall now examine how the decomposition in Theorem 6 extends to our Jordan domain  $\Omega$ .

**Theorem 30.** For  $\gamma, \delta \in C^\infty(\mathbb{T})$  with  $\gamma', \delta' \neq 0$  the following are equivalent

1. Every function in  $C^\infty(\partial\Omega)$  has a unique decomposition as the sum of a function in  $A^\infty(\Omega)$  and a function in  $A_0^\infty(\hat{\mathbb{C}} \setminus \overline{\Omega})$
2.  $C^\infty(\partial\Omega) = A^\infty(\Omega) \oplus A_0^\infty(\hat{\mathbb{C}} \setminus \overline{\Omega})$
3.  $C^\infty(\mathbb{T}) = A^\infty(D) \oplus [A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D}) \circ \delta^{-1} \circ \gamma]$
4.  $C^\infty(\mathbb{T}) = [A^\infty(D) \circ \gamma^{-1} \circ \delta] \oplus A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$

*Proof.* The first and second items are equivalent by Proposition 2. Let us now show the equivalence of the second and third items.

Let  $P : C^\infty(\partial\Omega) \rightarrow A^\infty(\Omega)$  be the projection corresponding to the splitting  $C^\infty(\partial\Omega) = A^\infty(\Omega) \oplus A_0^\infty(\hat{\mathbb{C}} \setminus \overline{\Omega})$ . Then  $Q : C^\infty(\mathbb{T}) \rightarrow A^\infty(D)$ ,  $Q(u) = P(f \circ \gamma^{-1}) \circ \gamma$ , is a projection as usual, hence  $C^\infty(\mathbb{T}) = A^\infty(D) \oplus \text{Ker} Q$ . We determine the kernel

$$\begin{aligned} Q(f) = 0 &\iff P(f \circ \gamma^{-1}) = 0 \iff \\ &f \circ \gamma^{-1} \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{\Omega}) \iff f \circ \gamma^{-1} \circ \delta \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D}) \iff \\ &f \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D}) \circ \delta^{-1} \circ \gamma \end{aligned}$$

The equivalence of the second and fourth items is similar.  $\square$

**Question 31.** Is there a Jordan domain  $\Omega$  and a function  $f \in C^\infty(\partial\Omega)$  that can't be decomposed as  $f = g + h$  for any  $g \in A^\infty(\Omega)$  and  $h \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{\Omega})$ ?

We suspect the answer is affirmative for the following reason: The map  $w(e^{i\theta}) = e^{-i\theta}$  is a welding by the conformal welding theorem and certainly,  $A^\infty(D) \circ w + A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$  is not a direct sum. But we don't yet know if the functions  $\gamma, \delta$  realizing the welding are  $C^\infty$  diffeomorphisms of  $\mathbb{T}$ .

The conformal welding theorem states in particular that every quasi-symmetry of the circle is a welding ([2],[10]). An injection  $f : A \rightarrow \mathbb{C}$ ,  $A \subseteq \mathbb{C}$ , is quasi-symmetric, if there is an increasing homeomorphism  $\eta : [0, +\infty) \rightarrow [0, +\infty)$  with

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq \eta \left( \frac{|x - y|}{|x - z|} \right)$$

for all triples  $x, y, z \in A$ ,  $x \neq z$ . Clearly, diffeomorphisms  $\mathbb{T} \rightarrow \mathbb{T}$  are quasi-symmetric.

## 5. INTERNALLY TANGENT CIRCLES

Take a disc  $D'$  in the interior of  $D$ , with  $\partial D'$  tangent at 1 to  $\mathbb{T} = \partial D$ , and let  $\Omega = D - \overline{D'}$ . We denote by  $\mathbb{T}$  the boundary of  $D$  (as usual) and the boundary of  $D'$  by  $\gamma$ . We examine whether or not every function  $f \in A^p(\Omega)$  has a decomposition as  $f = g + h$  for  $g \in A^p(D)$  and  $h \in A_0^p(\hat{\mathbb{C}} \setminus \overline{D'})$ . The case  $p = +\infty$  is easily dealt with:

**Theorem 32.** *Every function in  $A^\infty(\Omega)$  can be written uniquely as the sum of a function in  $A^\infty(D)$  and of a function in  $A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D'})$ .*

*Proof.* If  $f \in A^\infty(\Omega)$  then clearly,  $f \in C^\infty(\mathbb{T} - \{1\})$ . The derivatives of  $f$  extend continuously over  $\mathbb{T}$  hence  $f \in C^\infty(\mathbb{T})$ ; we similarly have that  $f \in C^\infty(\partial D')$ . By Theorem 6, there are  $g \in A^\infty(D)$  and  $h \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D})$  so that  $f = g + h$  on the unit circle. We extend  $h$  on  $D^c$  by setting  $h = f - g$  over  $D - D'$ ; the derivatives of  $h$  are also continuously extended this way. It is easy to see that the extended  $h$  is holomorphic over  $\overline{D'}^c$ , so  $h \in A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D'})$  as desired. This proves the existence of the decomposition. For the uniqueness of this decomposition, we observe that Liouville's Theorem implies that  $A^\infty(D) \cap A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D'}) = 0$ .  $\square$

Combining this with Proposition 2 we obtain:

**Corollary 33.**  $A^\infty(\Omega) = A^\infty(D) \oplus A_0^\infty(\hat{\mathbb{C}} \setminus \overline{D'})$ .

**Question 34.** For  $p < +\infty$ , is there an  $F \in A^p(\Omega)$  that can't be written as the sum of a function in  $A^p(D)$  and another in  $A_0^p(\hat{\mathbb{C}} \setminus \overline{D'})$ ?

We suspect that the decomposition does not hold (just as in the case for the circle/Jordan curve) and will now collect some sufficient conditions for a function  $F \in A^p(\Omega)$  to not be in  $A^p(D) + A_0^p(\hat{\mathbb{C}} \setminus \overline{D'})$ .

The Cauchy transform is defined as

$$C_{\mathbb{T}}(F) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Assume  $F = f + g$ ,  $f \in A^p(D)$  and  $g \in A_0^p(\hat{\mathbb{C}} \setminus \overline{D'})$ . Then  $C_{\mathbb{T}}(F) = C_{\mathbb{T}}(f) + C_{\mathbb{T}}(g)$ . We calculate

$$C_{\mathbb{T}}(F) = \begin{cases} f(z) & \text{if } z \in D \\ -g(z) & \text{if } z \notin D \end{cases}$$

If  $\gamma$  is the boundary of  $\partial D'$ , we can similarly prove that

$$C_{\gamma}(F) = \begin{cases} f(z) & \text{if } z \in D' \\ -g(z) & \text{if } z \notin D' \end{cases}$$

We have

$$\lim_{z \rightarrow 1, z \in D} C_{\mathbb{T}}(F) = \lim_{z \rightarrow 1, z \in \Omega} C_{\mathbb{T}}(F) = \lim_{z \rightarrow 1, z \in D'} C_{\gamma}(F) \in \mathbb{C}$$

and

$$\lim_{z \rightarrow 1, z \notin D} C_{\mathbb{T}}(F) = \lim_{z \rightarrow 1, z \notin D'} C_{\gamma}(F) = \lim_{z \rightarrow 1, z \in \Omega} C_{\gamma}(F) \in \mathbb{C}$$

**Question 35.** Let  $p < +\infty$ . Does there exist  $F \in A^p(\Omega)$  so that either one of the aforementioned limits doesn't exist, or a pair of limits exists but the limits are not equal? In particular, is there a function  $F \in A(D)$  such that  $\lim_{z \rightarrow 1, |z| < 1} C_{\mathbb{T}}(F)$  does not exist (in  $\mathbb{C}$ ) ?

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UNIVERSITY OF ATHENS, DEPARTMENT OF MATHEMATICS, 157 84 PANEPISTEMIOPOLEIS, ATHENS, GREECE  
 E-mail address: nikolaosgm@gmail.com